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History of Mathematics
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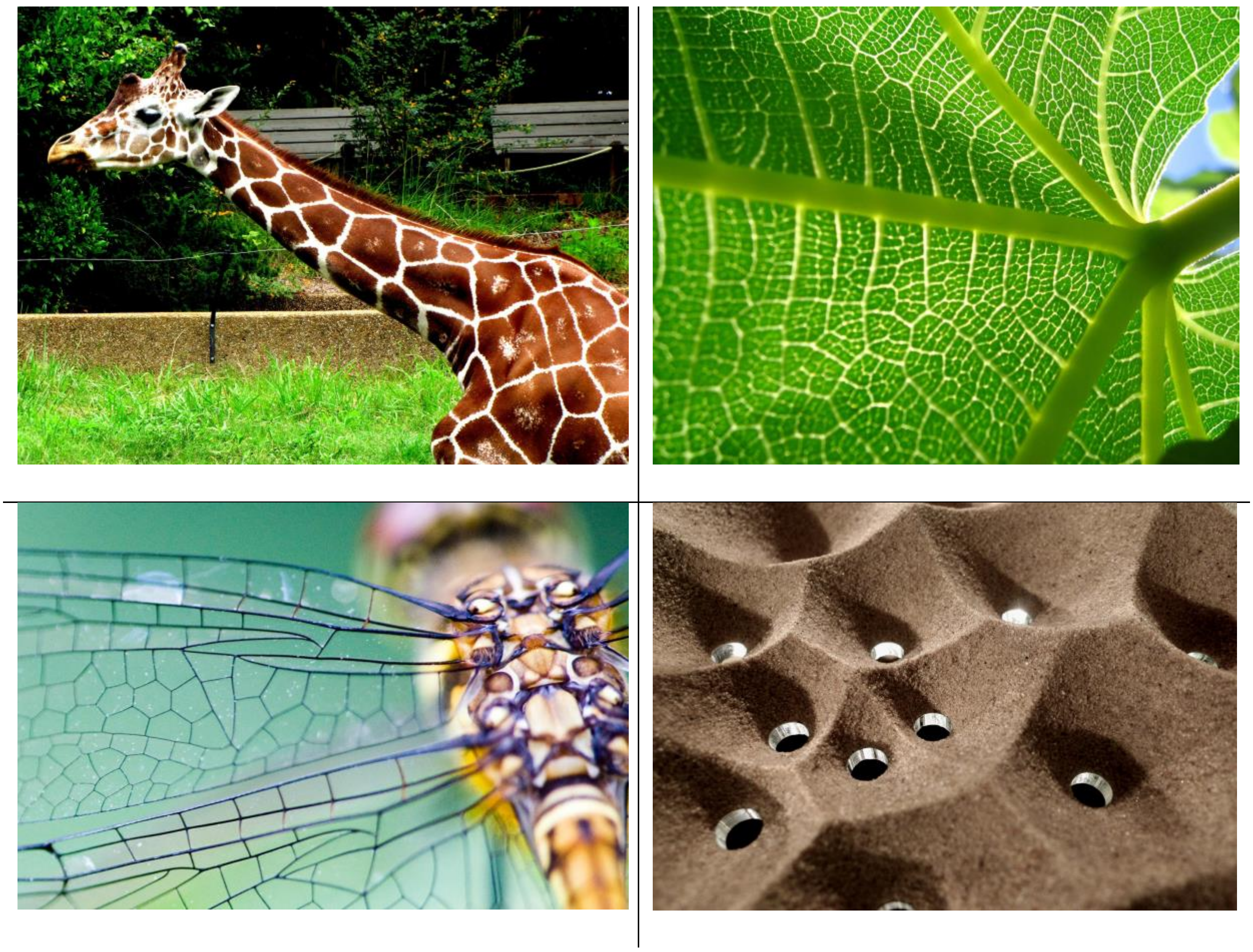
Voronoi Diagram

Mathematicians are simply great pattern seekers. We look around, we observe, we generalize, we deduce. What Mathematics has always been is a way to bring more harmony into one's life. To an outsider, however, math usually doesn't mean much. The phrase, 'I'm just not good at math,' became more of a cliché to most people. Nevertheless, they should not carry the blame for being so misguided. The problem lies much deeper - in the institution that claims so loudly to teach a subject, which should not be taught, but should rather be discovered. The institution prioritizes what it thinks is useful, as opposed to things that bring excitement and joy. This paper aims to bring you on a journey and help you connect the dots between the things you have already observed yourself.

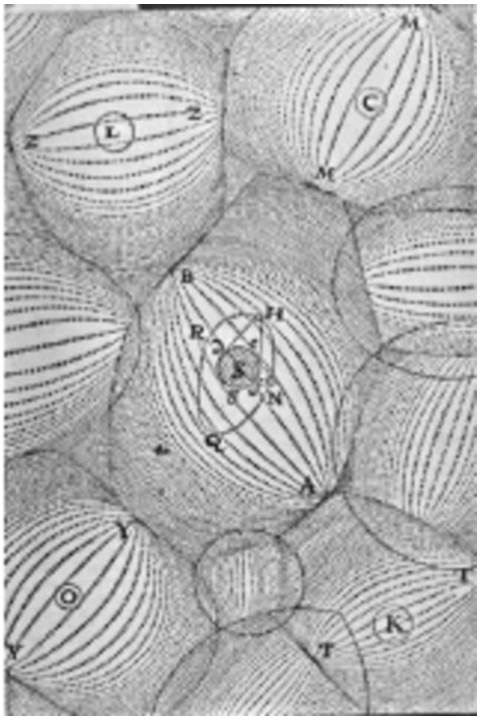
Concepts in math which are the most beautiful are the ones that are extremely simple, yet hidden from the common eye. The discovery of such concept is usually followed up by almost a barbaric roar, "Eureka!" Before I give an explicit definition of the topic of this paper, I want you to rather discover it for yourself.

Discovery

Look closely at 4 images shown below: a giraffe, a leaf, a dragonfly and a sand formation. After looking at them long enough it is hard not to admit that there is something uniting all 4 of them, some underlying principle. One can almost say that they are using the same set of formulas to achieve these different and unique, yet similar patterns. What is difficult, however, is to find this set of formulas, to prove that they are, in fact, connected. However, there are still some things one could deduce. Assuming that they are connected, one can theorize that the governing rule should not be overly complicated. Also, based on its appearance in living creatures (first 3 images), given our knowledge about evolutionary biology, it should be the case that the rule is not arbitrary, but rather has some usefulness. Lastly, we can also theorize that a mathematical model could in fact be applied, since sand formations are guided by simple physical laws.

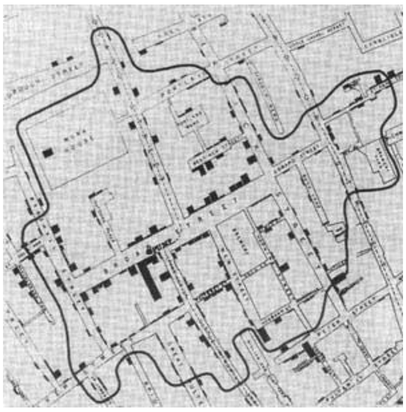


History



In order to understand how Voronoi Diagram was discovered, we should understand the set of interests and challenges encountered by the people of the past. The very first instance of Voronoi Diagram in literature can be found in "Principia Philosophiae" by R. Descartes, 1644. Being the religious man he was, he came up with an interesting hypothetical. Descartes imagined the universe to be a two-dimensional plane split into many different heavens. He drew several planets and then assumed that each planet has a heaven of its own. You might ask why? It really is not clear, but he might have been wondering which heaven he would be going to if he were to die while space travelling. The justification of this hypothetical is not that important, what is important is the conclusion he arrived to. It only makes sense to go to the heaven of the planet closest to the place of his death. It is obvious that if he was travelling next to the Earth, he should go to Earth's heaven, and if he was travelling next to Jupiter, he should go Jupiter's heaven, and so on. However, it becomes less clear if he was travelling somewhere in between many different planets. Thus, Descartes drew the above diagram showing the approximation of how the universe should be split given some arbitrary placed planets, which on the diagram are named as: S, L, C, K, O, and so on. There are some interesting details on his diagram. First, we see several circles and curvy edges. Famously, circle is a construction which helps find set of points equidistant from a given point, and since we are dealing with distances, it makes sense to include them into the diagram in some way (we will later show that while Descartes use of circles was incorrect, they still show up in a different form). We can also notice that while he correctly split most of the universe, there are some problematic regions in the corners, thus his method is a heuristic rather than an algorithm.

Moving away from hypotheticals, Voronoi Diagrams have very useful applications. One of the most notable early applicable appearances can be found in the work, "On chloroform and other anaesthetics : their action and administration" by John Snow. In 1854, a severe outbreak of cholera occurred on the streets of London, which was a part of a bigger 1846-1860 worldwide cholera pandemic. At the time, the cause of the infection was unknown. Luckily, a British physician, John Snow was ready to tackle the problem. He mapped out the deaths that occurred in 40 Broad Street as shown in the figure (one solid rectangle per death per household). John then noticed that a lot of the deaths very clustered in certain regions. He hypothesized the cause to have something to do with water, which lead him to also plotting water pumps of the region. John assigned to each region a water pump closest to it, since that's where most likely people would go to get their water supply. Result was astonishing, deaths nearly perfectly felt into the region of the Broad Street pump. He then used chlorine to clean the water, thus ending the outbreak. Nowadays, John Snow is known as the father of the epidemiology, an area of medicine which deals with the spreads and controls of diseases.

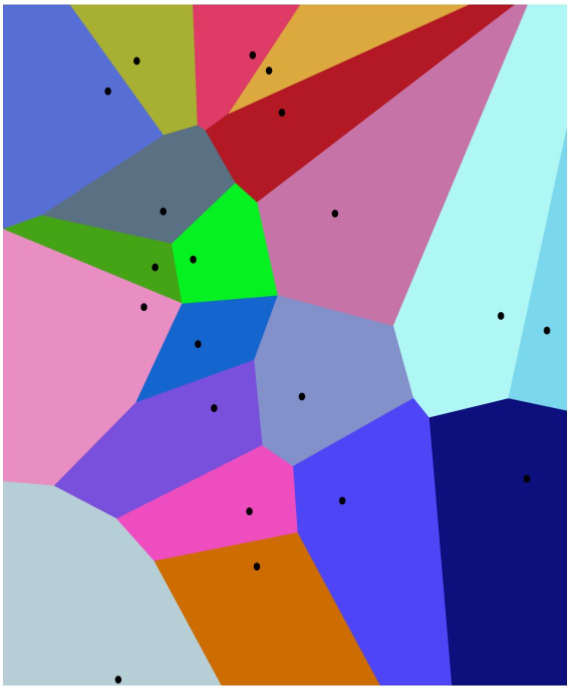


One can notice some interesting properties of his diagram. Unlike Descartes', Snow's diagram does not seem smooth. We see a lot of sharp corners, which require an explanation. What differentiates this problem from the last one is the our working space and the metric. Since we are dealing with distances, we have to double check the definitions. A distance between two points in the city is not the length of a straight line, because unfortunately, we cannot walk through the buildings. The distance we are looking for is similar to the Manhattan distance, since each path must go through streets and avenues. The reason why it is not exactly Manhattan distance is because we do not have a perfect city grid, which is why John had to manually find borderlines of each region, as there weren't any simple shortcuts.

Mathematics

There were two main contributors to the theory of Voronoi Diagrams. In 1850, a German mathematician Peter Gustav Lejeune Dirichlet used Voronoi Diagram in his studies of quadratic forms. In 1908, Russian mathematician Georgy Fedosievych Voronoi (from whom the diagram got its name) generalized and studied the problem in the n-dimensional case.

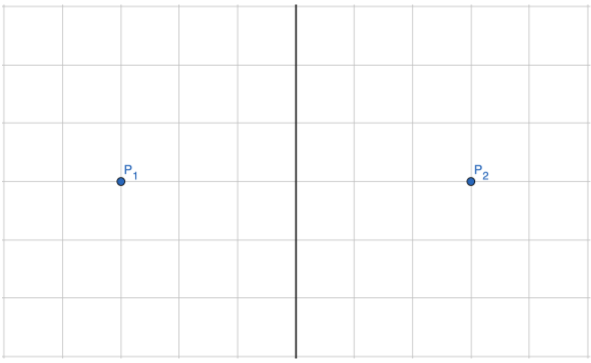
Let's now formally define the diagram. Let X be a metric space (which usually is a plane with a Euclidean metric like in the case with planets) and let $P = \{p_1, \ldots, p_n\}$ be a set of n points. We need to find a set of n Voronoi cells $R = \{R_1, \ldots, R_n\}$, where R_k is a set of points in X for which p_k is the closest point. It might sound overwhelming, but it actually is pretty straightforward. Think of the metric space, X as the universe, points as planets and Voronoi cells as Descartes' "heavens." The problem is - given a set of planets, find the shapes of their surrounding heavens.



An example of such problem can be seen on the diagram. We have our set of points, and our job is to paint the plane into different regions such that if we were to pick a random location we would know which point is closest to us based on the color of our region.

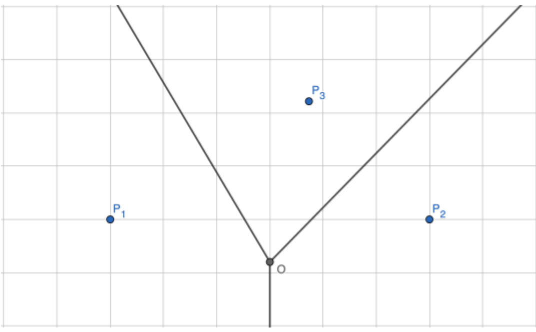
Before proving properties of this diagram, notice some details. First, if you were to choose a point on a borderline, it would have several closest points (so the soul would need to split). Second, all regions are convex, meaning the segment connecting any two points of the same region lies within the same region. Notice also that since our plane is infinite, some of the regions (ones on the sides) actually go forever. Finally, observe that all edges are straight. This will be true as long as we work with discrete points (but will change if generalize the problem to a set of curves, rather than discrete points).

Let's now consider simple cases. What would a Voronoi Diagram look like for a plane (which, with Euclidean metric, will be the default choice of X,) and $n = 2$? Since we only have 2 points, we need to divide the plane into 2 regions. What is easier to think of are the borderlines of the regions. We know that the points on our borderline are equally distant from p_1 and p_2. This should ring a bell, since it implies that we are looking for a perpendicular bisector of the segment p_1p_2. In the diagram, our perpendicular bisector is represented by a solid line. We then can see that for any point to the left of the line, the distance to p_1 would be shorter than to p_2, and for any point to the right, p_2 would be closer, therefore we solved the problem for $n = 2$.



It will be useful to introduce two definitions. Let $B(p,q)$ denote the perpendicular bisector between p, q, i.e. $B(p,q) = \{x \in R^2 \mid dist(p,x) = dist(x,q)\}$.. Notice that $B(p,q) = B(q,p)$, since it's a line between two points. Let $H(p,q)$ denote the half plane formed by a $B(q,p)$ which includes p, i.e. $H(p,q) = \{x \in R^2 \mid dist(p,x) < dist(x,q)\}$. Notice that, $H(p,q) \neq H(q,p)$ because first one is the left half plane, and second one is the right half plane.

Now add a third point p_3. Here is where the construction gets interesting. We want to find the Voronoi region of p_1, which we denote as R_1.



The trick is to realize that for any point x in R_1, $dist(x,p_1) < dist(x,p_2)$ and $dist(x,p_1) < dist(x,p_3)$, therefore $R_1 = H(p_1,p_2) \cap H(p_1,p_3)$. Similarly, $R_2 = H(p_2,p_1) \cap H(p_2,p_3)$ and $R_3 = H(p_3,p_1) \cap H(p_3,p_2)$. In fact, we can generalize this approach to any n. Given $P = \{p_1, \ldots, p_n\}$, we can find R_k as:

$$R_k = \bigcap_{p_i \neq p_k}^{p_i \in P} H(p_k, p_i)$$

This is amazing! We now have a theoretical way of finding Voronoi cells. Not only that, but we also gained some information about the borderlines, consisting of Voronoi edges and Voronoi vertices, which in the previous example were the point *O* and the 3 rays coming out of it. We know that Voronoi edges fully lie in the set B consisting of all possible perpendicular bisectors of the given set of points. In addition, the set of Voronoi vertices is a subset of all intersections of elements in B. Just based on these simple observation we can deduce certain properties.

Definition: A subset *S* is *convex* if for any two points $p, q \in S$, the segment $pq \subset S$.

Lemma 1. Voronoi cells are convex.

Proof

Under this definition it is clear that Half-planes are convex. Further, notice that given two convex regions A and B, the intersection $A \cap B$ is also convex. This is true because

$\forall p, q \in A \cap B$ we have $pq \subset A$ and $pq \subset B$ therefore $pq \subset A \cap B$

Thus $R_k = \bigcap_{p_i \neq p_k}^{p_i \in P} H(p_k, p_i)$, i.e. intersection of finitely many convex half-planes, thus convex.

Definition: A circle, *C* is empty with respect to *P* if it does not contain any of its points

Definition: A Voronoi circle, C_{ijk} is an empty circle passing through points p_i, p_j, p_k

Lemma 2. Point *v* is a Voronoi vertex iff it is the center of a Voronoi circle.

Intuition

Looking at the *n = 3* case one can observe that point *O* is equally far from all 3 points because it lies on their perpendicular bisectors implying that $dist(O, p_1) = dist(O, p_2) = dist(O, p_3)$. Therefore, we can construct a circle at *O* which would pass through all 3 points.

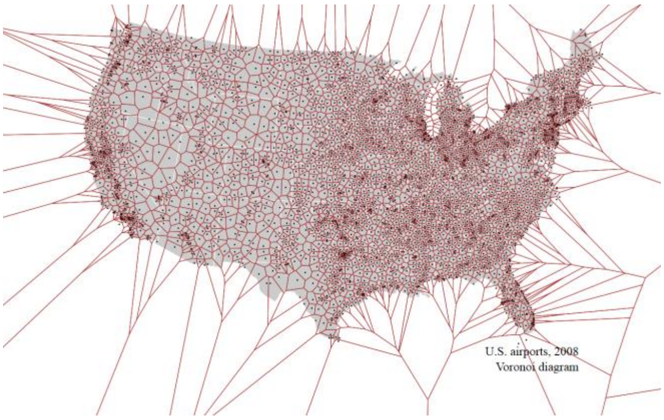
Proof

=> Let *v* be a vertex and $p_1, p_2, p_3, \ldots p_k$ be the set of points whose Voronoi cells touch *v* (usually it's only 3 points, but could be more). We know that $d(v, p_i)$ is the same for all p_i, thus they all lie on the same circle centered at *v*. On the other hand, any other point from *P* lies further by definition, thus outside the circle, therefore the chosen circle is a Voronoi circle.

<= Given a Voronoi circle *Cijk* we know that the distance from the center, v of the circle is the same to at least 3 points p_i, p_j, p_k, since it's an empty circle, it is also the shortest distance, therefore, v is the Voronoi vertex.

Perspectives

We now know some properties of Voronoi diagram. However, it is somewhat troublesome to actually construct such a diagram. We have shown that each Voronoi cell can be constructed by intersecting *n-1* of its half-planes. This method does not seem to be the most optimal one, which is why it might be beneficial to get a different perspective.



By now, we looked at the problem form a perspective of a random point in space looking around trying to figure out the closest "planet". An example of where such vision can be useful can be seen in emergency plane landings. What pilots always want to know is where is the closest airport located at any time of their flight. It would be rather terrifying if such information was not immediately available. It sounds like we need a Voronoi diagram of the U.S. with our set of points being all airports. In this case, our algorithm seems totally valid, because we would only need to compute Voronoi diagram ones in a while, since airports do not move. Well, even with airports, not all of them are available at all times. Each has a very tight schedule, has a list of planes getting ready to fly off, and another queue of planes waiting to land. Hopefully, this shows some motivation for why we would want to improve our methods.

Regardless, we can shift our perspective just to see how the same problem can arise in a different setting. A different perspective is usually brought by contemporary mathematicians W.P.Thurston, L. Paul Chew and others.

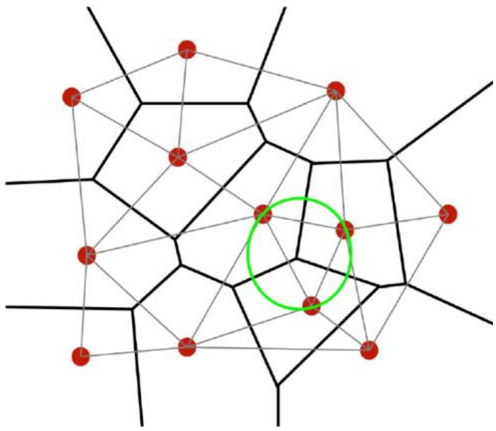
Imagine looking at the problem from "planet's" perspective. As a planet, you know that because all other planets are so far, the close atmosphere definitely belongs to your Voronoi cell. So goes for any other planet. Next step is to expand your Voronoi cell a little more simultaneously with all other planets. And a little more and so on. A the territory once claimed by one planet, cannot be claimed by another because the first one got there first. If all planets were expanding their Voronoi cell circularly and with the same speed, then eventually we would get our wanted Voronoi diagram! Another way to visualize the process is as such: imagine dropping 2 drops of different paint at two places on the table. Initially, the table will be clean except for the 2 small areas occupied by drops. However, as time would go on, drops would spread circularly in all direction eventually painting the whole table. Or imagine a perfectly flat sand surface lying on a flat surface with small point wise gates. Imagine then what would happen if the gates were all open at the same time. First, the sand right above the gates would fall, then the sand right next to the gates and so on causing the chain reaction. Imagining such a process might be challenging, but likely you do not have to because you have already seen the result in the very beginning of the paper (page 2).

Under this new definition, one can imagine constructing Voronoi diagram with a computer simulation by locating circles with initial zero radius and a small constant rate of growth. This idea of introducing a "flow" comes up frequently in mathematics, which means that one could apply a whole new set of tools when solving related problems.

This approach is rather interesting. Geologists and meteorologists use Voronoi diagrams (which they call Thiessen polygons, named after American meteorologist Alfred H. Thiessen) in order to conduct analysis of weather and climate. An interesting scenario that one could think of are movements of earth plates and earthquakes. They start off at a point and then gradually expand in a way similar to the one we described. One could also think of the interactions of big cyclones, which (to a rough approximation) create their own Voronoi diagrams.

One might be wondering is there any other mathematical concept that can be associated with Voronoi diagram, and it turns out there is. Georgy Voronoi was a doctoral advisor of Boris Delaunay (or Delone). Delaunay's biography is truly fascinating. His father, Nikolai Borisovich Delone was a professor of physics in St. Petersburg University, which gave Boris an excellent education. According to many of his teacher, Boris was good not only at STEM fields, but also at music and art. Nevertheless, his love for exact fields was higher. At the age of 15, he turned his room into a physics laboratory, and even constructed a bronze telescope. Boris also came up with a unique proof for Gauss's reciprocity law. How much more could he do? In university, he organized an aeronautic club, and became of the first glider pilots. Boris Delaunay not only was a great Soviet mathematician, but was also one of the first soviet rock climbers. In 1913, he was considered to be in the top 3 best Russian climbers, and later in his life even got a mountain named after him. Delaunay has contributed to not only modern geometry, but also to mathematical crystallography (study of crystal structures), and Galois theory, a topic on modern algebra. Later on, he found the first mathematical Olympiad in USSR, which set the tone for competitive mathematics of the Soviet Union, and then spent much of his time teaching (O'Connor, Robertson. "Boris Nikolaevich Delone")

Given all that, we will only focus on one of his contributions known as Delaunay triangulation, which relates closely to our Voronoi Diagram. Again, let $P = \{p_1, \ldots, p_n\}$ be a set of n. We then construct the Voronoi diagram. Notice now that every edge or the diagram lies in between two points. We can use this idea to construct a mathematical graph. The vertices of this graph are our points $P = \{p_1, \ldots, p_n\}$. Two vertices are connected by an edge if there is an edge in the Voronoi diagram between them. An example is shown on the figure. We notice that the resulting graph is a triangulation, we call such a triangulation, a Delaunay triangulation.



Definition: Delaunay triangulation of a set of points, *P* denoted as DT(P) is a triangulation such that the circumference of any of its triangles doesn't include any of its points.

For example, if we refer to the previous diagram, we can see that for any chosen triangle of the triangulation, no other vertex is inside the corresponding circle (ex. green circle). This construction might seem strange, but it should ring some bells, because we actually already did something similar, i.e. Voronoi circles.

Analogously, any two points whose Voronoi cells share an edge are connected by a Delaunay edge. We will omit the formal proof, which can be found on page 8 of "*Notes on Convex Sets*" by Jean Gallier, but the essential idea is to look at the Voronoi circles and use *Lemma 2* to show that no two Delaunay edges can intersect, except at vertex.

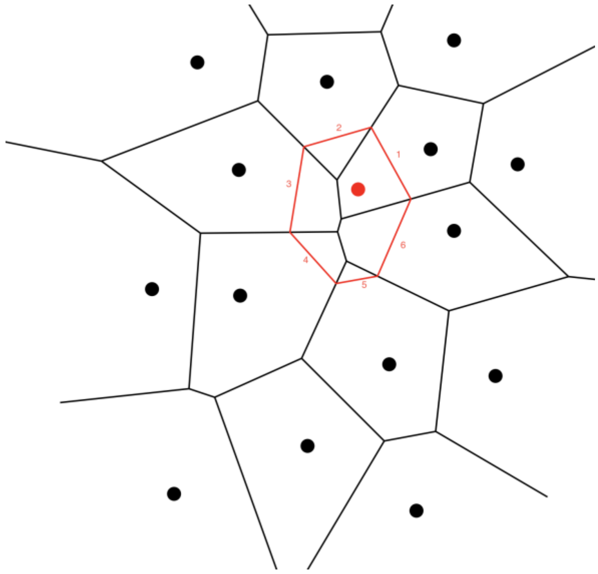
In fact, one can prove that the Delaunay triangulation is the dual graph of the Voronoi diagram meaning. Duality turns out to be an important mathematical concept. In short, the dual graph of a give planar graph *G* is a graph that has a vertex in each face of the original graph, and an edge whenever two faces share an edge. While in general, dual graphs are not unique, the Delaunay triangulation is a unique dual graph of a Voronoi diagram (given a small additional requirement that no four points are co-circular).

Constructions

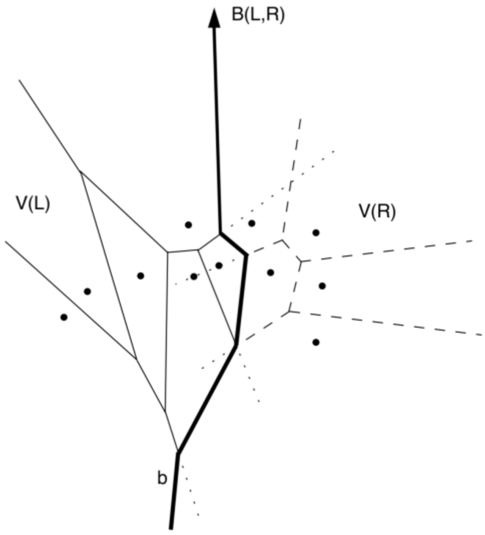
Now that we discussed the basic definitions and properties of the diagram, we will talk about how to practically construct it.

First, we can look at the algorithm already discussed earlier, known as the "Naïve" algorithm. We can construct each Voronoi cell by intersecting the half-planes. There are several problems with it. For one, such an algorithm gives little to no insight into the set of vertices and edges. In addition, it takes a very long time to run, since we need to not only compute all possible half-planes but to also intersect them for each cell. There are also precision problems that arise with it related to the fact that we do not actually look for edges directly.

Second algorithm is known as an "incremental" one. Not only this algorithm is faster and more precise, it is also most useful for personal drawing, or "doodling." This algorithm is recursive, meaning we will use the Voronoi diagram for n points, to construct the diagram for n+1 points (n old points and 1 new point). We are going to modify the existing Voronoi diagram for n points, denote as V_n. Let's add the n+1th point (red point). We then find the Voronoi cell of the V_n, call R_1 which includes the new point. Since the new point is in the cell, we know that part of the region must belong to the new cell, so we draw the perpendicular bisector, labeled as *1*. This bisector intersect the edge of the cell R_2, therefore part of its cell also should belong to the new region, so we again draw the perpendicular bisector between p_{n+1} and p_2, and so on until we finished the loop and come back to region R_1. Thus what we did is we only looked at half planes between the new point and the closest points. Since each step had to make at each iteration we make at most *1, 2, .., n* comparisons, the overall complexity is on the order of n^2 (as opposed to naïve's $n^2 \log n$).



It turns out that there are algorithms much faster, which have a linear complexity *O(n log n)*, such as "Divide and Conquer" described in *Voronoi Diagrams* by Franz and Klein. Reader can also refer to images in *Algorithms for Constructing Voronoi Diagrams* by Sacrist´an. Readers which are familiar with sorting algorithms will notice that it has the same complexity as the most efficient sorts, such as for example the quick sort. The idea is very similar, we solve the problem recursively by dividing *n* points into two piles of roughly *n/2* points (Left and Right), constructing their Voronoi diagrams (V(L) and (V(R)) and merging them (with B(L,R)).



Conclusion and thoughts

In the beginning of the paper we looked at the geometric patterns and we looked for potential rules that could derive these shapes. Hopefully by now, these constructions became more intuitive. For example, such pigmentation in giraffe's skin might point to the fact the process of pigmentation actually relates to discrete point sources which gradually spread, in the same way as described by Thurston and Chen. Maybe the reason why dragonfly's wings look as the do is due to the fact that the biology was searching for construction that minimizes distances, similar to how John Snow was looking at the regions minimizing walking distances till water towers.

The beauty of mathematics comes not from its complexity, but rather from its simplicity, and Voronoi diagram is an perfect example of how simple principles rule most complex systems.

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